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Local Clebsch parametrization of Beltrami equilibira

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A Beltrami field is a vector field $\boldsymbol{w}$ that is aligned with its own curl,

$$\nabla \times \boldsymbol{w} = \hat{h}\,\boldsymbol{w} \quad \text{in} \quad \Omega. \tag{1}$$

Here, Ω is a smoothly bounded domain in $\mathbb{R}^3$, and the vector field $\boldsymbol{w} \in C^\infty(\Omega)$ is assumed smooth. Evidently, a Beltrami field is an eigenvector of the curl operator with eigenvalue $\hat{h}$. In this study, we shall be concerned with nontrivial Beltrami fields, i.e. Beltrami fields with finite helicity density $h = \boldsymbol{w} \cdot \nabla \times \boldsymbol{w} = w^2 \hat{h} \neq 0$ in the whole Ω. The eigenvalue $\hat{h}$ is called the *proportionality factor* (or coefficient).

Beltrami fields occur as steady solutions of the Euler equations of fluid dynamics, and as force free magnetic fields in ideal magnetohydrodynamics. Furthermore, they can be used to construct time-decaying solutions of the Navier-Stokes and magnetohydrodynamics equations. When one leaves the fluid formulation in favor of the particle picture, Beltrami fields define a class of antisymmetric operators that dictate particle dynamics by acting on a Hamiltonian function, and enable a formulation of statistical mechanics even in the absence of canonical phase space [1-2].

In most physical applications, solenoidal solutions of (1) are desirable,

$$\nabla \cdot \boldsymbol{w} = 0 \quad \text{in} \quad \Omega. \tag{2}$$

If $\boldsymbol{w}$ is a fluid velocity, equation (2) is a consequence of the continuity equation in a regime of constant fluid density. When $\boldsymbol{w}$ describes a magnetic field, (2) represents its intrinsic divergence-free nature. In addition, boundary conditions may be required,

$$\boldsymbol{w} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega, \tag{3}$$

with $\boldsymbol{n}$ the unit outward normal to the boundary $\partial\Omega$.

While Beltrami fields play important roles in various theories, the existence of regular nontrivial solutions of the Beltrami equation (1) is a mathematical challenge. Yoshida and Giga [3] proved general existence of strong solutions of system (1-3) for constant proportionality factors $\hat{h} \in \mathbb{C}$, and revealed an interesting relation between the spectral structure and the topology (homology) of the domain. The nonconstant-$\hat{h}$ Beltrami equation becomes a "nonlinear" elliptic-hyperbolic system, and the general theory is not known. Recently, Enciso and Peralta-Salas [4] showed that a global solution may exist under some "rare" condition.

The aim of this study is to show the existence of local solutions to the nonconstant-$\hat{h}$ Beltrami equation (1). The divergence free condition (2) and boundary conditions (3) are omitted at first, and the solution is found in a small neighborhood of a point in the domain of interest.

First, by application of the Lie–Darboux theorem of differential geometry, we establish a local representation (Clebsh-like parametrization) theorem for Beltrami fields [5]. We find that, locally, a Beltrami field has a standard form corresponding to an Arnold–Beltrami–Childress flow with two of the parameters set to zero: for every $\boldsymbol{x} \in \Omega$ there exists a neighborhood $U \subset \Omega$ of $\boldsymbol{x}$ and smooth curvilinear coordinates $(\ell, \psi, \theta) \in C^\infty(U)$ such that

$$\boldsymbol{w} = \cos\theta\,\nabla\psi + \sin\theta\,\nabla\ell \quad \text{in} \quad U. \tag{5}$$

Furthermore, a Beltrami flow admits two local invariants, the coordinate θ representing the physical plane of the flow, and the angular momentum-like quantity in the direction across the plane $L_\theta = \ell\cos\theta - \psi\sin\theta$. As a consequence of the theorem, we then derive a method to construct Beltrami fields with given proportionality factor. This method, based on the solution of the eikonal equation, guarantees the existence of Beltrami fields for any orthogonal coordinate system such that at least two scale factors are equal. In other words, one must find an orthogonal coordinate system (ℓ, ψ, θ) such that

$$|\nabla\theta| = |\hat{h}|, \quad |\nabla\ell| = |\nabla\psi| \quad \text{in} \quad \Omega. \tag{6}$$

Then, a solution of (1) can be given in the form of (5). Such solution is solenoidal provided that $\Delta\ell = \Delta\psi = 0$ throughout Ω.

Finally, by application of the local representation theory, we construct families of solenoidal and non-solenoidal Beltrami fields with both constant and non-constant $\hat{h}$. Boundary conditions (3), the generalization of the local theory to larger classes of fluid equilibria, and symmetry properties of solutions are also discussed.

References

[1] N Sato and Z Yoshida 2018 *Diffusion with finite-helicity field tensor: A mechanism of generating heterogeneity* Phys. Rev. E **97** 022145

[2] N Sato 2019 *Beltrami operators and their application to constrained diffusion in Beltrami fields* J. Phys. A: Math. Theor. **52** 235203

[3] Z Yoshida and Y Giga 1990 *Remarks on spectra of operator rot* Math Z. **204** pp 235-245

[4] A Enciso and D Peralta-Salas 2016 *Beltrami fields with a nonconstant proportionality factor are rare* Arch. Ration. Mech. Anal. **220** pp 243-260

[5] N Sato and M Yamada 2019 *Local representation and construction of Beltrami fields* Physica D: Nonlinear Phenomena **391** pp 8-16