



Time-dependent relaxed magnetohydrodynamics — inclusion of cross helicity constraint using phase-space action

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Taylor's magnetic *energy*-minimization principle [1] yields static plasma equilibria that are compatible with ideal MHD, yet, by removing the ideal “frozen-in flux” constraint, non-ideal reconnection is allowed to occur in the *relaxation* process leading to these equilibria.

A dynamical generalization of this idea via an *action*-based formulation [2] of Multiregion Relaxed MHD (MRxMHD) in 2015 included Taylor relaxation of the magnetic field \mathbf{B} but did not incorporate a relaxation model for the fluid. Nor did it include coupling between \mathbf{B} and Eulerian fluid velocity \mathbf{u} except at the interfaces between the multiple relaxation subregions Ω_i .

A previous attempt [3] at relaxing \mathbf{u} using a seemingly analogous Lagrangian to that used in [2] for \mathbf{B} led to results inconsistent with energy relaxation. In this paper we show that a new “phase-space” Lagrangian,

$$L_{\Omega}^{\text{Rx}} = \int_{\Omega} \rho \mathbf{u} \cdot \mathbf{v} dV - W_{\Omega}^{\text{Rx}}, \quad (1)$$

gives Euler–Lagrange equations *consistent* with previous work on relaxed (Rx) steady-flow MHD equilibria [4], and appears to give a satisfactory generalization of the relaxation concept to dynamics on time scales on the order of or longer than relaxation times.

Above, $\Omega \in \{\Omega_i\}$, ρ is mass density with ideal variation $\delta\rho = -\nabla \cdot (\rho \xi)$ under Lagrangian fluid displacements ξ , and \mathbf{v} is a reference field “conjugate” to \mathbf{u} , varying as a Lagrangian-constrained velocity, $\delta\mathbf{v} = \partial_t \xi + \mathbf{v} \cdot \nabla \xi - \xi \cdot \nabla \mathbf{v}$, whereas the relaxed velocity field \mathbf{u} is freely variable except on the interface $\partial\Omega$ (as are \mathbf{B} and pressure p).

In the last term of Eq. (1), the “Hamiltonian” W_{Ω}^{Rx} is the total (kinetic plus magnetic) plasma energy in Ω , with additional Lagrange multiplier terms to constrain total magnetic helicity (as in [1]), entropy [2], and *cross helicity* [4] to couple \mathbf{B} and \mathbf{u} .

The Euler–Lagrange equations, necessary conditions for the first variation of the action integral $\int L_{\Omega}^{\text{Rx}} dt$ to vanish under the variations prescribed above, are

$$p = \tau_{\Omega} \rho, \quad (2)$$

$$\rho \mathbf{v} = \rho \mathbf{u} - \nu_{\Omega} \mathbf{B} / \mu_0, \quad (3)$$

$$\nabla \times \mathbf{B} = \mu_{\Omega} \mathbf{B} + \nu_{\Omega} \mathbf{u}, \quad (4)$$

$$\text{and } \partial_t \mathbf{u} + (\nabla \times \mathbf{u}) \times \mathbf{v} = -\nabla h \quad (5)$$

where τ_{Ω} is the entropy Lagrange multiplier (temperature in eV divided by ion mass = square of isothermal sound speed squared) and ν_{Ω} and μ_{Ω} are the cross-helicity and magnetic-helicity Lagrange multipliers, respectively, whereas μ_0 is the vacuum

permeability. Also, the *Bernoulli head* h is defined as

$$h = \frac{u^2}{2} + \tau_{\Omega} \ln \frac{\rho}{\rho_{\Omega}}, \quad (6)$$

where ρ_{Ω} is a non-dimensionalizing spatial constant.

Note that Eq. (3) allows us to eliminate \mathbf{v} in favor of \mathbf{u} . However \mathbf{v} does play a crucial role as, unlike [4], we restrict variations of ρ to conserve mass microscopically under both Lagrangian displacements and time evolution, i.e. ρ obeys the continuity equation $\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$. As Eq. (3) implies $\nabla \cdot (\rho \mathbf{u}) = \nabla \cdot (\rho \mathbf{v})$, the Eulerian flow \mathbf{u} also respects continuity,

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (7)$$

thus completing our formulation of relaxed dynamics within a single subregion Ω_i .

Assuming the interfaces between contiguous subregions act as massless ideal-MHD transport barriers, our phase space action principal also reproduces the coupling relation derived in previous formulations of MRxMHD (e.g. [2]), namely the continuity of $p + B^2/2\mu_0$ across interfaces, thus completing our new dynamical MRxMHD formulation.

As indicated above, this formulation is completely consistent with Finn and Anderson's [4] axisymmetric relaxed equilibrium, which respects the ideal Ohm's Law equilibrium requirement that $\nabla \times (\mathbf{u} \times \mathbf{B}) = 0$. The status of this requirement in general, time-dependent problems will be discussed, and applications to problems such as normal mode stability studies will be indicated.

Numerical studies will be discussed elsewhere in this conference (Qu *et al.*)

References

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