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Deformation of Lie-Poisson algebras producing chirality

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1. Outline

Linearization of a Hamiltonian system around an energy-Casimir equilibrium point yields a linear Hamiltonian system, which has the Hamiltonian spectral symmetry [1]: If $\lambda = \gamma + \mathrm{i}\,\omega$ is an eigenvalue, $-\lambda$ is also an eigenvalue (moreover λ^* is also an eigenvalue). However, linearization around a singular equilibrium point works out differently, and spectral symmetry breaking occurs, resulting in chiral dynamics. This interesting phenomenon was first found in analyzing the chiral motion of a rattleback, a boat-shaped top having misaligned axes of inertia and geometry [2]. To elucidate how non-Hamiltonian (or chiral) spectra are generated, we study the three-dimensional Bianchi Lie-Poisson systems and classify the prototypes of singularities that causes chirality (the rattleback model is a class-B, type IV Lie-Poisson system). The central idea is the deformation of the underlying Lie algebra; we show that the class-B algebras (by Bianchi's classification), which are produced by asymmetric deformations of a simple algebra, yield chiral spectra when linearized around the singularities.

2. Lie-Poisson algebra

There is a systematic method for constructing Poisson brackets from any given Lie algebra. Let X be a vector space, on which we define a Lie algebra with a bracket [,]. The adjoint action $\mathrm{ad}_h = [\ ,h]$ represents dynamics in X. The dual space X^* is the phase space, which is the totality of real-valued linear functionals representing observables. We denote $\langle\ ,\ \rangle: X \times X^* \to \mathbf{R}$, and define the coadjoint action $\mathrm{ad}_h^* = [\ ,h]^*$, where $[\ ,\ ]^*: X \times X^* \to X^*$ is given by

$$\langle [g,h],\phi\rangle = \langle g,[h,\phi]^*\rangle.$$

We call X^* a Poisson manifold, and consider $\mathrm{C}^\infty(X^*)$, the space of smooth real-valued functionals on X^*. For $F(\phi) \in \mathrm{C}^\infty(X^*)$, we denote its gradient by $\partial_\phi F \in X$. Then,

$$\{G,H\} = \langle [\partial_\phi G, \partial_\phi H],\phi\rangle = \langle \partial_\phi G, [\partial_\phi H,\phi]^*\rangle$$

is a Poisson bracket (Jacobi's identity inherits from [,]). Hamilton's equation is $\dot{G} = \{G,H\}$ (H is a Hamiltonian). For $\dot{G} = \langle \partial_\phi G, \dot{\phi}\rangle$, Hamilton's equation implies

$$\dot{\phi} = J(\phi)\partial_\phi H, \qquad (1)$$

We call $J(\phi) = [\circ,\phi]^*$ a Poisson operator.

See [2] for the complete list of Poisson operators corresponding to the Bianchi classification of three-dimensional Lie-Poisson algebras.

3. Deformation

Physically, $\langle \mathrm{ad}_h \circ, \phi\rangle = \langle [\circ,h],\phi\rangle = \langle \circ, [h,\phi]^*\rangle$ means the observation of the action of a Hamiltonian vector h by an observable ϕ. When we transform ϕ by $M \in \mathrm{End}(X^*)$, the dynamics (adjoint action) will look different. Of course, there is a constraint on M so that the modified dynamics maintains to be Hamiltonian. Explicitly, we may write $\langle [g,h], M\phi\rangle = \langle M^t[g,h],\phi\rangle$, so the deformed bracket $[\,,]_M = M^t[\,,]$ must be a Lie bracket. We have the following observations for three-dimensional Lie algebras:

(1) Starting from the simple algebra $[e_j,e_k]_{\mathrm{IX}} = \epsilon_{jk\ell}e_\ell$ (i.e. so(3) algebra, which is called type-IX in Bianchi's table), we can produce all possible Lie algebras with some M. The deformed Poisson operator is $J(\phi) = [\circ, M\phi]^*_{\mathrm{IX}} = (\circ \times M\phi)$.
(2) Every symmetric M produces a class-A algebra.
(3) If rank M =3, only symmetric M produces a Lie algebra.
(4) If rank $M \le 2$, any M such that $M = N \oplus 0$ produces a Lie algebra. When N is not symmetric, then the deformed algebra is class-B.

4. Chiral spectra of class-B Lie-Poisson system

By the Lie-Darboux theorem, a 3-dimensional Poisson manifold is foliated by a Casimir (which is a functional $C \in \mathrm{C}^\infty(X^*)$ such that $\{C,H\} = 0\ (\forall H)$); see [2] for the complete list of Casimirs. If M is a symmetric matrix (i.e. class-A), $C = \langle M^t\phi,\phi\rangle/2$ is the Casimir. Otherwise (i.e. class-B), the Casimir is a more complicated function (the leaves are not algebraic varieties).

Because the Poisson operator $J(\phi)$ is a linear function of ϕ, it has a *singularity* where rank $J(\phi)$ drops to zero. When $M = N \oplus 0$, the ϕ_3-axis (the line of $\phi_1 = \phi_2 = 0$, which we denote by Σ) is the singularity. For Hamilton's equation (1), the singularity Σ is an equilibrium point (i.e. $\dot{\phi} = 0$). Linearizing (1) around Σ, we obtain (denoting the perturbation by $\tilde{\phi}$, and $\partial_\phi H|_\Sigma = h$),

$$\dot{\tilde{\phi}} = J(\tilde{\phi})h = h \times M\,\tilde{\phi}\,, \qquad (2)$$

which is a Hamiltonian system only if M is symmetric (i.e., class-A). Then, the Casimir $C = \langle M^t\tilde{\phi},\tilde{\phi}\rangle/2$ plays the role of Hamiltonian. For M of class-B, (2) is not Hamiltonian, so that the Krein symmetry is broken.

References

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[2] Z. Yoshida, P.J. Morrison, and T. Tokieda, Phys. Lett. A **381** (2017), 2772-2777.