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The partition of enstrophy between zonal and turbulent components

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1. Outline

We consider barotropic flow on a beta-plane, and study the partition of the enstrophy between the zonal (ordered) and wavy (turbulent) components of the vorticity; we derive an estimate of the minimum of the zonal enstrophy. The conservations of the energy and angular momentum pose constraints on the minimization of the zonal enstrophy. The result is useful to evaluate the strength of zonal flow in terms of macroscopic constants of motion.

The problem has a mathematical nontriviality in that the target functional, the zonal enstrophy, is not coercive in the topology of the phase space of vortex dynamics. We invoke a singular perturbation to elucidate the mini/max property of the critical points. The result is compared with numerical simulation; the theory provides us with a perspective to interpret a zonal flow as an eigenstate belonging to a quantum number = number of zones.

2. Formulation

2.1. Vorticity on a beta-plane

We consider barotropic flow on a beta-plane:

$$\Omega = \{\mathbf{z} = (x,y)^T \,; x \in [0,1), y \in [0,1)\}.$$

Here, x is the longitude and y is the latitude. The state variable is the fluid vorticity ω. We define the Gauss potential φ by $-\Delta\varphi = \omega$ or $\varphi = K\omega$ $(K = (-\Delta)^{-1})$. On x=0 and 1, we impose periodix boundary condition, while on y=0 and 1, Dirichlet boundary condition $\varphi|_{y=0} = \varphi|_{y=1} = 0$

Taking into account the Coriolis force, the governing equation of ω is

$$\partial_t\omega + \{\omega + \beta y, \varphi\} = 0, \tag{1}$$

where $\{f,g\} = (\partial_x f)(\partial_y g) - (\partial_x g)(\partial_y f)$, and β is a real constant number measuring the meridional variation of the Coriolis force. We use the following functionals which are constants of motion of (1).

Enstrophy: $Q(\omega) = \frac{1}{2}\|\omega\|^2$

Energy: $E(\omega) = \frac{1}{2}\langle\varphi, \omega\rangle$

We denote $\langle f,g\rangle = \int_\Omega f(z)g(z)\,d^2z$ and $\|f\| = \langle f,f\rangle$.

2.2. Zonal and wavy components

We say ω is zonal when $\partial_x\omega \equiv 0$ in Ω and define zonal component of ω as $\omega_z := \int \omega\, dx$. Also we define wavy component as $\omega_w := \omega - \omega_z$.

3. Zonal enstrophy

Here, we take into account the constraint by energy and seek the minimizer of

$$Z_\varepsilon(\omega) - \mu E(\omega). \tag{2}$$

Here, we denote $Z_\varepsilon(\omega) = Q(\omega_z) + \varepsilon Q(\omega_w)$, which is zonal enstrophy containing singular perturbation term, and μ is a Lagrange multiplier.

The Eular-Lagrange equation is

$$\omega_z + \varepsilon\omega_w - \mu K\omega = 0. \tag{3}$$

The solution satisfying the boundary conditions is

$$\varphi = A\sin\lambda y + B\sin\lambda_2 x \sin\lambda_3 y, \tag{4}$$

which gives

$$\omega = A\lambda^2 \sin\lambda y + B\frac{\mu}{\varepsilon}\sin\lambda_2 x\sin\lambda_3 y, \tag{5}$$

where $\lambda = \sqrt{\mu}$, ${\lambda_2}^2 + {\lambda_3}^2 = \frac{\mu}{\varepsilon}$.

The energy constraint $E(\omega) = C_E$ reads

$$C_E = \frac{A^2\lambda^2}{4} + \frac{B^2\mu}{8\varepsilon}. \tag{6}$$

Because the wavy enstrophy $Q(\omega_w) = B^2\frac{\mu^2}{\varepsilon^2}$ must be finite, we have to demand $\lim_{\varepsilon\to 0}\frac{B^2}{\varepsilon} = 0$.

Then, A^2, determined by (6) for given C_E, increases with $\varepsilon \to 0$. We obtain

$$\lim_{\varepsilon\to 0} Q(\omega_Z) = C_E\lambda^2, \tag{7}$$

which evaluates the local maximum of the critical zonal enstrophy that occurs when all energy is given to the zonal component. Figure1 shows a result of numerical simulation. The actual zonal enstrophy is slightly smaller than the theoretical value of (7), because finite ω_w remains.

4. Conclusion

We have studied the partition of enstrophy between zonal and wavy components. The difference form earlier works [1][2] is in that we consider the conservation of energy as the constraint, instead of Casimirs. As the self-organization of zonal flows is a spontaneous process driven by internal energy, the amount of energy poses an essential constraint. On the other hand, the relaxation into zonal flow requires topological changes, so that the Casimir constraints must be removed. Numerical simulation supports our theoretical framework.

References

[1] T.Shepherd, J. Fluid Mech. 196, 291-322 (1988)

[2] K.Ishioka and S.Yoden, J. Metrorological Soc. Japan 74 167-174 (1966)

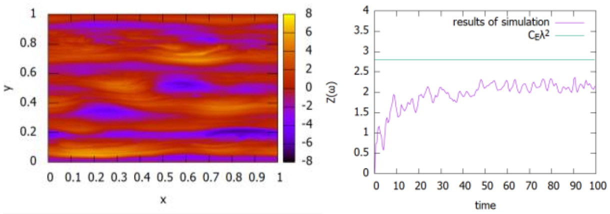

Figure1 (left) Zonal flow self-organized in simulation. (right) Comparison of zonal enstrophy between result of simulation and theoretical estimate. Zonal enstrophy in simulation is smaller than the theoretical estimate because a part of energy is given to the wavy component.