3rd Asia-Pacific Conference on Plasma Physics, 4-8,11.2019, Hefei, China

# Time-dependent relaxed magnetohydrodynamics — inclusion of cross helicity constraint using phase-space action

Robert L. Dewar[1], Joshua W. Burby[2], Zhisong Qu[1], Naoki Sato[3]

[1] Mathematical Sciences Institute, Australian National University, [2] Los Alamos National Laboratory, USA, [3] Research Institute for Mathematical Sciences, Kyoto University, Japan

robert.dewar@anu.edu.au

Taylor's magnetic *energy*-minimization principle [1] yields static plasma equilibria that are compatible with ideal MHD, yet, by removing the ideal "frozen-in flux" constraint, non-ideal reconnection is allowed to occur in the *relaxation* process leading to these equilibria.

A dynamical generalization of this idea via an *action*-based formulation [2] of Multiregion Relaxed MHD (MRxMHD) in 2015 included Taylor relaxation of the magnetic field **B** but did not incorporate a relaxation model for the fluid. Nor did it include coupling between **B** and Eulerian fluid velocity **u** except at the interfaces between the multiple relaxation subregions $\Omega_i$.

A previous attempt [3] at relaxing **u** using a seemingly analogous Lagrangian to that used in [2] for **B** led to results inconsistent with energy relaxation. In this paper we show that a new "phase-space" Lagrangian,

$$L_\Omega^{\mathrm{Rx}} = \int_\Omega \rho \mathbf{u}\cdot\mathbf{v}\, dV - W_\Omega^{\mathrm{Rx}}, \qquad (1)$$

gives Euler–Lagrange equations *consistent* with previous work on relaxed (Rx) steady-flow MHD equilibria [4], and appears to give a satisfactory generalization of the relaxation concept to dynamics on time scales on the order of or longer than relaxation times.

Above, $\Omega \in \{\Omega_i\}$, $\rho$ is mass density with ideal variation $\delta\rho = -\nabla\cdot(\rho\boldsymbol{\xi})$ under Lagrangian fluid displacements $\boldsymbol{\xi}$, and $\mathbf{v}$ is a reference field "conjugate" to **u**, varying as a Lagrangian-constrained velocity, $\delta\mathbf{v} = \partial_t\boldsymbol{\xi} + \mathbf{v}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\mathbf{v}$, whereas the relaxed velocity field $\mathbf{u}$ is freely variable except on the interface $\partial\Omega$ (as are $\mathbf{B}$ and pressure $p$).

In the last term of Eq. (1), the "Hamiltonian" $W_\Omega^{\mathrm{Rx}}$ is the total (kinetic plus magnetic) plasma energy in $\Omega$, with additional Lagrange multiplier terms to constrain total magnetic helicity (as in [1]), entropy [2], and *cross helicity* [4] to couple **B** and **u**.

The Euler–Lagrange equations, necessary conditions for the first variation of the action integral $\int L_\Omega^{\mathrm{Rx}} dt$ to vanish under the variations prescribed above, are

$$p = \tau_\Omega \rho, \qquad (2)$$

$$\rho\mathbf{v} = \rho\mathbf{u} - \nu_\Omega \mathbf{B}/\mu_0\,, \qquad (3)$$

$$\nabla\times\mathbf{B} = \mu_\Omega\mathbf{B} + \nu_\Omega\mathbf{u}, \qquad (4)$$

and

$$\partial_t\mathbf{u} + (\nabla\times\mathbf{u})\times\mathbf{v} = -\nabla h \qquad (5)$$

where $\tau_\Omega$ is the entropy Lagrange multiplier (temperature in eV divided by ion mass = square of isothermal sound speed squared) and $\nu_\Omega$ and $\mu_\Omega$ are the cross-helicity and magnetic-helicity Lagrange multipliers, respectively, whereas $\mu_0$ is the vacuum permeability. Also, the *Bernoulli head* $h$ is defined as

$$h = \frac{u^2}{2} + \tau_\Omega \ln\frac{\rho}{\rho_\Omega}\,, \qquad (6)$$

where $\rho_\Omega$ is a non-dimensionalizing spatial constant.

Note that Eq. (3) allows us to eliminate **v** in favor of **u**. However **v** does play a crucial role as, unlike [4], we restrict variations of $\rho$ to conserve mass microscopically under both Lagrangian displacements and time evolution, i.e. $\rho$ obeys the continuity equation $\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0$. As Eq. (3) implies $\nabla\cdot(\rho\mathbf{u}) = \nabla\cdot(\rho\mathbf{v})$, the Eulerian flow **u** also respects continuity,

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0, \qquad (7)$$

thus completing our formulation of relaxed dynamics within a single subregion $\Omega_i$.

Assuming the interfaces between contiguous subregions act as massless ideal-MHD transport barriers, our phase space action principal also reproduces the coupling relation derived in previous formulations of MRxMHD (e.g. [2]), namely the continuity of $p + B^2/2\mu_0$ across interfaces, thus completing our new dynamical MRxMHD formulation.

As indicated above, this formulation is completely consistent with Finn and Anderson's [4] axisymmetric relaxed equilibrium, which respects the ideal Ohm's Law equilibrium requirement that $\nabla\times(\mathbf{u}\times\mathbf{B}) = 0$. The status of this requirement in general, time-dependent problems will be discussed, and applications to problems such as normal mode stability studies will be indicated.

Numerical studies will be discussed elsewhere in this conference (Qu *et al.*)

**References**

[1] J.B. Taylor, Rev. Mod. Phys. **58**, 741–63 (1986)
[2] R.L. Dewar, Z. Yoshida, A. Bhattacharjee and S.R. Hudson, J. Plasma Phys. **81**, 515810604-1–22 (2015)
[3] N. Sato and R.L. Dewar, arxiv:1708.06193 (2017)
[4] J.M. Finn and T.M. Antonsen, Jr., Phys. Fluids **26**, 3540–52 (1983)

Some of this work was supported by the Australian Research Council grant DP170102606. Also by the National Science Foundation grant DMS-1440140 while RLD and JWB were in residence at the Mathematical Sciences Research Institute in Berkeley in 2019. This presentation was also partially supported by a grant from the Simons Foundation/SFARI (560651, AB).