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Nambu bracket and induced Lie-Poisson bracket for ideal MHD equation

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The motion of fluid particles are regarded as transformation by the diffeomorphism group of a region filled by a fluid. Denote the velocity and the vorticity fields by $\boldsymbol{v}$ and $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v}$ as functions of the space $\boldsymbol{x}$ and the time t. The *circulation* and the *helicity*

$$h[\boldsymbol{v}]=\frac{1}{2}\int \boldsymbol{\omega} \cdot \boldsymbol{v}\mathrm{dV}$$

are particular integral invariants of an ideal fluid in the sense that they are conserved under any diffeomorphism, independently of the Euler equations, unlike the energy, the impulse and the angular impulse. The helicity is an integral that makes distinction between `right-handed' and `left-handed' of the vortical structure, and holds the key to an understanding of the mechanism for formation of large-scale coherent structures in the atmosphere, oceans and the universe.

When the viscosity, the heat conductivity and the electric resistivity are ignored, the motion of a neutral and a conducting fluids constitutes the Hamiltonian dynamical system of infinite degrees of freedom. The equation is the Lie-Poisson equation with degenerate Hamiltonian structure [1], rather than the canonical Hamiltonian equation. In the configuration space, symplectic leaves form a foliation structure, with each leaf specified by the value of the Casimir invariants. The motion of the fluid does not go through the entire space of the observables, but is restricted to a symplectic leaf. Among the Casimirs is the helicity.

We can confirm, with use of the Lie-Poisson equation, that the Casimir invariants are constant of motion for any choice of the Hamiltonian function, but they do not make their appearance. The Nambu bracket [2], which is an extension of the Lie-Poisson brackets, makes them explicit; in the Nambu bracket, the Casimir invariants are given the role of the Hamiltonian functions. This is the case with the helicity for a barotropic fluid [3]. The purpose of this investigation is to extend the Nambu bracket to the magnetohydrodynamics (MHD).

As a preliminary step, we describe the Nambu-bracket representation for an incompressible barotropic fluid. The density ρ of the fluid is taken to be constant The Euler equation is written in the form of the Poisson equation,

$$\frac{dF}{dt} = \{F, H\},$$

for a functional F of $\boldsymbol{v}$ and the Hamiltonian $H = \frac{\rho}{2}\int v^2 dV$ with use of the Li-Poisson bracket [1],

$$\{F, H\} = \int (\nabla \times \boldsymbol{v}) \cdot \left(\frac{\delta F}{\delta \boldsymbol{v}} \times \frac{\delta H}{\delta \boldsymbol{v}}\right) dV.$$

The first factor in the integrand is the vorticity and is given by the functional derivative of the helicity h,

$$\{F, H\} = \int \frac{\delta h}{\delta \boldsymbol{v}} \cdot \left(\frac{\delta F}{\delta \boldsymbol{v}} \times \frac{\delta H}{\delta \boldsymbol{v}}\right) dV = \{F, H, h\}_{vvv}.$$

This is a prototype of the Nambu bracket [3].

For MHD, the Lorentz force stands as an obstacle against the conservation of the circulation and the helicity. In order to seek all the topological invariants, we resort to Noether's theorem, which states that, in the framework of Hamilton's principle of the least action, a symmetry keeping the action brings in the conservation law and *vice versa*. Specifically, we inquire into the *particle labeling symmetry* behind the topological invariant. By applying this idea to the variational principle for the momentum equation of the MHD, we gain the *cross-helicity*, as an integral invariant (cf. [4, 5]),

$$h_c[\boldsymbol{v}, \boldsymbol{D}] = \int \boldsymbol{v} \cdot \boldsymbol{D}\mathrm{dV},$$

where $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{x}, t)$ is a solenoidal vector field frozen into the fluid,

$$\frac{D}{Dt}\left(\frac{\boldsymbol{D}}{\rho}\right) = \left(\frac{\boldsymbol{D}}{\rho} \cdot \nabla\right)\boldsymbol{v},$$

with additional constraints required by the density stratification and the Lorentz force,

$$\nabla \cdot \boldsymbol{D} = 0,\ (\mathbf{D} \cdot \nabla)\mathrm{s} = 0,\ \nabla \times \left(\boldsymbol{B} \times \frac{\boldsymbol{D}}{\rho}\right) = \mathbf{0}.$$

Here ρ, s and $\boldsymbol{B}$ are the density, the specific entropy of the fluid and the magnetic field. For the ideal MHD, the total mass M and the total entropy S defined by

$$M = \int \rho \mathrm{dV},\ S = \int \rho \mathrm{s}\mathrm{dV},$$

and the *magnetic helicity* h_{m}, defined with use of the vector potential $\boldsymbol{A}$ for the magnetic field $(\boldsymbol{B} = \nabla \times \boldsymbol{A})$,

$$h_{\mathrm{m}}[\boldsymbol{A}] = \frac{1}{2}\int \boldsymbol{A} \cdot \boldsymbol{B}\mathrm{dV},$$

are also Casimir invariants. The Nambu bracket for the ideal MHD is constructed using all the four Casimir invariants h_c, h_m, M and *S*.

It remains as a difficulty to show that the cross-helicity h_{c} is a Casimir invariant [6]. This difficulty is resolved by our Nambu bracket. The Lie-Poisson bracket induced from the Nambu bracket is not the well-known one [1] itself, but gives an extension of this. The extended Lie-Poisson bracket automatically guarantees the cross-helicity to be a Casimir invariant. A remark is given to Noether's second theorem.

References

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