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Relaxed MHD with local and global constraints

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Ideal magnetohydrodynamics (IMHD) is the compressible Euler fluid dynamics of a velocity field **u** interacting via the **j**X**B** force with the subdynamics of electric and magnetic fields **E** and **B** coupled by Maxwell's equations (without displacement current)s and the Ideal Ohm's Law (IOL) **E**+**u**X**B** = 0.

Each fluid element is subject to microscopic constraints that combine to forbid topological changes (reconnection) in the configuration of magnetic field lines during IMHD evolution. Where reconnection would release magnetic energy, it becomes energetically favourable for localized gradients to grow and become large enough that small nonideal effects can overcome this topological barrier and cause magnetic islands to open up. In 3-D toroidal magnetic confinement devices such as stellarators, a fractal chaotic hierarchy of magnetic islands can develop that would be prohibitively expensive to simulate using higher-order, multi-scale equations that add dissipation, e.g. resistive MHD.

In 1958 Woltjer [1] enunciated an energy- relaxation-under-constraints principle (which he attributed largely to Chandrasekhar) as a way of avoiding such simulations: *The states of stable equilibrium which a given hydromagnetic system may attain are the states in which the energy has a minimum value compatible with the integrals of the equations of motion of the system.* (The integrals Woltjer lists as constraints are the magnetic helicity, cross helicity, angular momentum and total mass.)

Later Taylor [2] used a simplified version of the Woltjer principle to analyze the results of the Zeta reversed-field pinch experiment, and invoked plasma turbulence as the relaxation mechanism. An attractive feature of the Taylor relaxation variational principle is that its Euler–Lagrange equation is a simple PDE (the linear-force-free or *Beltrami* equation. This provides a resolution of the long-standing mathematical problem regarding the existence of IMHD equilibria in 3-D toroidal plasmas by regularizing away the singularities that arise if the magnetic field lines are constrained to lie on smoothly nested invariant tori (magnetic surfaces). Instead, because the Beltrami equation is elliptic, solving it requires no assumptions as to the detailed behaviour of magnetic field lines, so magnetic islands and chaos cause no problems. Unfortunately, this at the expense of no pressure gradient, making Taylor states fusion-*irrelevant*. This prompted Bhattacharjee and Dewar [3] to modify the magnetic helicity integral by weighting **A.B** with a weight function that is a function only of the flux function, giving an ideal invariant in 2-D plasmas. To handle 3-D problems, Hudson et al [5] developed instead the stepped-pressure equilibrium code SPEC, using *multiple* Taylor states (Multiregion Relaxed MHD or MRxMHD).

Hamilton's Principle has been used [5,6] to make Relaxed MHD truly *dynamical*, with Lagrangians incorporating constraints of magnetic and cross helicity. However this formulation does not respect the IOL, potentially violating the desideratum that all equilibrium solutions of RxMHD form a subset of all IMHD equilibria. A recent paper [7] lays the formal groundwork for rectifying this deficiency. In order to impose a weak form of the IOL constraint on RxMHD two forms of the iterative augmented Lagrangian penalty function method are proposed and discussed. It is conjectured this weak-form regularization will allow reconnection and thus avoid the formation of the singularities that plague three-dimensional IMHD equilibria. A unified dynamical formalism is developed that can treat a number of MHD versions. Euler–Lagrange equations and a gauge-invariant momentum equation in conservation form are derived, in which the IOL constraint contributes an external force and internal stress terms until convergence is achieved.

References

I1] Woltjer, L. 1958a *A theorem on force-free magnetic fields*. Proc. Nat. Acad. Sci. (U.S.) **44**, 489–491; 1958b On Hydromagnetic Equilibrium. Proc. Nat. Acad. Sci. (U.S.) **44**, 833–841

[2] Taylor, J. B. 1974, *Relaxation of Toroidal Plasma and Generation of Reverse Magnetic Fields*. Phys. Rev. Lett. **33**, 1139–1141.; 1986 *Relaxation and magnetic reconnection in plasmas*. Rev. Mod. Phys. **58**, 741–763.

[3] Bhattacharjee, A. & Dewar, R. L. 1982 *Energy principle with global invariants*. Phys. Fluids **25**, 887–897.

[4] Hudson, S. R., Dewar, R. L., Dennis, G., Hole, M. J., McGann, M., von Nessi, G. & Lazerson, S. 2012 *Computation of multi-region relaxed magneto- hydrodynamic equilibria*. Phys. Plasmas 19, 112502

[5] Dewar, R. L., Yoshida, Z., Bhattacharjee, A. & Hudson, S. R. 2015 *Variational formulation of relaxed and multi-region relaxed magnetohydrodynamics.* J. Plasma Phys. **81**, 515810604.

[6] Dewar, R. L., Burby, J. W., Qu, Z. S., Sato, N. & Hole, M. J. 2020 Time-dependent relaxed magnetohydrodynamics – inclusion of cross helicity constraint using phase-space action. Phys. Plasmas **27**, 062504–1–22.

[7] Dewar, R. L., Burby, J. W., Qu, Z. S. 2021 Relaxed MHD with Ideal Ohm's Law Constraint. http://arxiv.org/abs/2106.12348