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Modification of Lie transform for the system with a fast-varying coordinate

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The Lie transform perturbation theory of Hamiltonian systems has been widely used in the field of plasma physics, for example, for studying the guided center motion of charged particles in a magnetic field and developing the nonlinear gyrokinetic theory, etc. Nowadays, plasma simulation has become an important tool, which leads the Lie transform perturbation theory receive further attention. This is because, unlike the standard gyrokinetic theory, the Lie transform formulation preserves Liouville's theorem and the energy conservation law for time-independent systems. These properties are especially important for long-period simulations. Besides, the Lie transform is a near identity transformation. It allows the expansion generator to be determined in the order-by-order analyses. Also, the backward transformation can be obtained rather easily from the forward transformation. The developments are extensively reviewed recently for example in Refs. [1] and [2], as well as in the book in Ref. [3].

In this work, we show that the conventional Lie transform perturbation theory needs to be modified when considering a system with a fast-varying coordinate for ordering consistency [4,3]. For the guiding center motion of a charged particle in a magnetic field, the gyrophase is this type of coordinates. This is related to the discrepancy of the results obtained by the direct expansion method and the Lie transform formulation. The direct method gives the Lagrangian to the first order as follows [2]

$$\begin{aligned} {}^{d}\Gamma &= \left(\frac{e}{mc}\boldsymbol{A} + u\boldsymbol{b}\right) \cdot d\boldsymbol{X} + \frac{mc}{e}\mu d\zeta \\ &\quad - \left(\frac{u^2}{2} + \mu B + \frac{e}{m}\varphi\right) dt, \end{aligned} \tag{1}$$

while the conventional Lie transform theory yields in the same order [5,6]

$${}^{d}\Gamma = \left(\frac{e}{mc}\boldsymbol{A} + u\boldsymbol{b}\right) \cdot d\boldsymbol{X} - \left(\frac{u^2}{2} + \mu B + \frac{e}{m}\varphi\right) dt. \tag{2}$$

Here, u is parallel velocity, μ is magnetic moment, $\boldsymbol{X}$ is the guiding center position, and ζ denotes the gyrophase. Comparing Eqs. (1) and (2) one can see that the gyromotion contribution "$(mc/e)\mu d\zeta$" is missing in the Lie trnsform formulation, although it is of the same order as the bounce motion contribution "$u\boldsymbol{b} \cdot d\boldsymbol{X}$".

As shown in [4], two reasons cause this discrepancy. First, the ordering difference between the temporal variation of gyrophase and that of the other phase space coordinates needs to be taken into account. Second, it is also important to note that the limit and derivative cannot be commuted in general. When these facts are taken into account, the near identity transformation rule for one form related to the Lagrangian becomes

$$\begin{aligned} (L_g\Gamma)_\mu &= g^\nu \left(\partial_\nu \gamma_\mu - \partial_\mu \gamma_\nu\right) \\ &\quad - g^\nu \left[\left(\partial_\nu \gamma_\delta\right)\left(\partial_\mu g_1^\delta\right) - \left(\partial_\mu \gamma_\delta\right)\left(\partial_\nu g_1^\delta\right)\right]. \end{aligned}$$

Here, γ_μ and Γ_μ are the one form Lagrangians before and after the near-identity transformation, and g^ν is the near-identity transformation generator. The contribution of the second term on the right is new. When coupled with the fast-varying coordinate $d\zeta$, the second term on the right gives rise to the otherwise missed term $(mc/e)\mu d\zeta$, which is actually of the same order as that of the first term and can only be obtained in the tedious higher-order perturbation analyses in the conventional theory. This resolves the discrepancy between the direct and Lie transform treatments in the Lagrangian perturbation theory for charged particle motion in a magnetic field.

The correction to the near identity transformation rule pointed out in this work [4] affects generally the Lie transform framework in the classical mechanics for the system with a fast-varying coordinate. Especially, it modifies the derivation of the nonlinear gyrokinetic equations, which will be addressed as well.

References

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