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On the Grad Conjecture in Anisotropic MHD

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The Grad conjecture represents a fundamental unsolved mathematical problem in the theory of PDEs: whether one can find isotropic magnetohydrodynamic equilibria or steady Euler flows that are not invariant under some continuous Euclidean isometry (according to the standard formulation of the Grad conjecture, the answer should be negative). This problem has broad implications for plasma physics and fluid mechanics, especially for the design of stellarators, where the existence of asymmetric confining magnetic fields remains a core theoretical issue.

In this study, we consider a weaker version of this mathematical problem: can one find a vector field such that both the vector field and its curl are tangential to a given set of nested toroidal surfaces? We show that the answer is always positive, and that these configurations can be identified with magnetohydrodynamic equilibria with anisotropic pressure [1].

In formulae, the problem considered here amounts to solving the system of PDEs,

$$[(\nabla \times \boldsymbol{w}) \times \boldsymbol{w}] \times \nabla\Psi = \boldsymbol{0}, \quad \nabla \cdot \boldsymbol{w} = 0, \qquad (1)$$

where $\Psi(\boldsymbol{x})$ is a given (flux) function whose level sets define a family of nested toroidal surfaces. Notice that standard isotropic MHD equilibria are solutions of (1), although the converse is not true.

By using a tailored Clebsch parametrization of the solution, we show that system (1) can be translated into the problem of determining a periodic solution with periodic derivatives of a two-dimensional linear elliptic second-order partial differential equation on each toroidal surface and prove the existence of smooth solutions. Examples of smooth solutions foliated by toroidal surfaces that are not invariant under continuous Euclidean isometries are also constructed explicitly, and they are identified as equilibria of anisotropic magnetohydrodynamics. These solutions have the following form:

$$\boldsymbol{w} = f(\Psi_\epsilon)\boldsymbol{\xi}_\epsilon, \qquad (2)$$

where f is any function of the flux function

$$\Psi_\epsilon = \frac{1}{2}\left\{\left[re^{-\epsilon Q(x,y)} - r_0\right]^2 + z^2 e^{-\epsilon S(z)}\right\}, \qquad (3)$$

and $\boldsymbol{\xi}_\epsilon$ the vector field

$$\boldsymbol{\xi}_\epsilon = \nabla\left(\varphi + \epsilon P(x,y)\right) \qquad (4)$$

with (r,φ,z) cylindrical coordinates, ϵ a real constant, $P(x,y)$ and $Q(x,y)$ harmonic conjugate functions in two dimensions, and $S(z)$ any function of z. We remark that the family of vector fields (2) includes steady solutions of anisotropic magnetohydrodynamics

$$(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} = \nabla \cdot \Pi, \quad \nabla \cdot \boldsymbol{w} = 0, \qquad (5)$$

which are not invariant under continuous (as well as discrete) Euclidean isometries, i.e. combinations of translations, rotations, and reflections. Here, Π is the anisotropic pressure tensor

$$\Pi^{ij} = \left(P - \frac{1}{2}\gamma\boldsymbol{w}^2\right)\delta^{ij} + \gamma w^i w^j, \quad i,j = 1,2,3. \qquad (6)$$

In particular, notice that (2) solves (5) with $P = 0$ and $\gamma = 1 - f^{-2}$. An example of anisotropic MHD equilibrium without continuous Euclidean isometries is given in figure 1 below.

In conclusion, this work shows that as long as pressure anisotropy is allowed steady smooth solutions of magnetohydrodynamics exist without continuous or discrete Euclidean isometries.

References

[1] N. Sato and M. Yamada, "*Nested invariant tori foliating a vector field and its curl: toward MHD equilibria and steady Euler flows in toroidal domains without continuous Euclidean isometries*," arXiv:2211.10757 (2022).

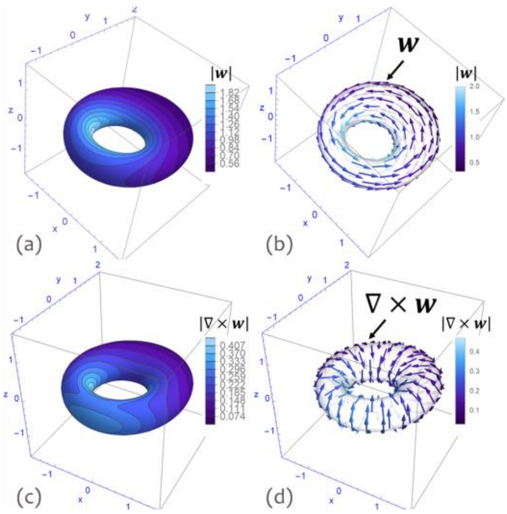

Figure 1: Contour plot (a) and vector plot (b) of an anisotropic MHD equilibrium $\boldsymbol{w}$ without continuous Euclidean isometries. Contour plot (c) and vector plot (d) of $\nabla \times \boldsymbol{w}$. See [1] for additional details.