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Clebsch representation and generalized enstrophy for relativistic plasma

K. Nunotani[1] and Z. Yoshida[1,2]
[1] Graduate School of Frontier Sciences, The University of Tokyo
[2] National Institute for Fusion Science
e-mail (speaker): nunotani.keiichiro20@ae.k.u-tokyo.ac.jp

The theory of relativistic plasmas is useful to model high-energy astronomical objects. In this context, topological constraints built in the governing equations play an essential role in characterizing structures self-organized by plasmas. Among various invariants of ideal models, the circulation is one of the most fundamental quantities, being included in other invariants like the helicity. The conventional enstrophy in a two-dimensional (2D) flow is defined as

$$\int_{\Sigma(t)} f\left(\frac{\omega}{\rho}\right) \rho \, dx^2 \qquad (1)$$

with a particle number density ρ, the scalar vorticity ω, an arbitrary smooth function f: $\mathbb{R} \to \mathbb{R}$, and an arbitrary region $\Sigma(t) \subset \mathbb{R}^2$ that co-moves with the 2D fluid. Enstrophy (1) is known to be constant and to measure the number of vortex filaments penetrating the surface $\Sigma(t)$.

By introducing potential fields φ, λ^1, σ_1, λ^2, and σ_2, the velocity field V of a fluid can represented as

$$mV := \nabla\varphi + \lambda^1\nabla\sigma_1 + \lambda^2\nabla\sigma_2. \qquad (2)$$

This expression is called a Clebsch representation of V, and the potential fields above are called Clebsch parameters or Clebsch variables [1]. Mathematically, a general three-dimensional (3D) vector field V can always be cast into the form of equation (2) [2]. The corresponding vorticity vector ω can be written as

$$\omega = \nabla\lambda^1 \times \nabla\sigma_1 + \nabla\lambda^2 \times \nabla\sigma_2. \qquad (3)$$

By invoking Clebsch variables, the conventional enstrophy in a 2D flow can be generalized to a 3D flow: for $k = 1, 2$,

$$\tilde{Q}_k(t) := \int_{\Omega(t)} f(\vartheta_k) \, \rho \, dx^3 \, , \qquad \vartheta_k := \frac{\omega_k \cdot \nabla\sigma_{3-k}}{\rho} \qquad (4)$$

with $\omega_k := \nabla\lambda^k \times \nabla\sigma_k$. The integration domain $\Omega(t) \subset \mathbb{R}^3$ is an arbitrary region that co-moves with the 3D fluid. Both $\tilde{Q}_1(t)$ and $\tilde{Q}_2(t)$ can be interpreted as topological charges of a 3D fluid element, which essentially measure circulation [3,4].

Since relativistic effects impart space-time coupling into the metric, such invariants are no longer constant in a relativistic setting. In particular, the non-relativistic enstrophy is no longer conserved in a relativistic plasma, implying that the conservation of circulation is violated.

In this work [5], we extend the (non-relativistic) enstrophy in a 3D flow to a Lorentz covariant form. To this end, we first derive relativistic fluid equations for the Clebsch parameters by using the principle of least action. We then show that the time evolution of the conventional enstrophy $Q_k(t)$ is given by:

$$\frac{d}{dt}Q_k(t) = \int_{\Omega(t)} \{ \, f'(\vartheta) \; c \; J(\log\gamma \, , \lambda^k, \sigma_k, \sigma_{3-k}) - f(\vartheta) \; n \; u^\mu \partial_\mu(\log\gamma) \, \} \; d^3x \qquad (5)$$

where $J(\xi^0, \xi^1, \xi^2, \xi^3)$ is the Jacobian determinant. Here dQ_k/dt is proportional to the derivative of $\log\gamma$, implying that $Q_k(t)$ is not preserved in a relativistic setting. Finally, we derive a relativistic enstrophy $\mathfrak{Q}_k(s)$ that is preserved by the relativistic fluid equations. An explanation of the geometric meaning carried by $\mathfrak{Q}_k(s)$ is given in figure 1. We also discuss simple examples and show that highly symmetric flows may preserve both $Q_k(t)$ and $\mathfrak{Q}_k(s)$ due to the vanishing of $u^\mu\partial_\mu(\log\gamma)$.

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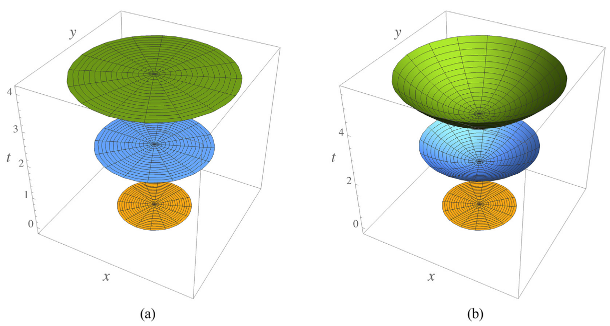

Figure 1. A conceptual drawing of t-plane (left) and s-plane (right) of the fluid flowing outward radially from the origin. The t-plane is a hyperplane in space-time at constant time t and used for the integral domain of the semi-relativistic enstrophy $Q_1(t)$. On the other hand, the s-plane is a curved hypersurface in space-time reflecting the intrinsic time s of the fluid and used for the integral domain of the relativistic enstrophy $\mathfrak{Q}_1(s)$. In this figure, the fluid farther from the origin is faster, so more time t has passed on the outer side [5].