Comparison between the generalized Grad-Shafranov equation for MHD equilibrium with flow and its reduced models

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Axisymmetric plasma equilibrium with flow in magnetohydrodynamics (MHD) is described by the generalized Grad-Shafranov (GGS) equation [1]. It is the generalized equation of the Grad-Shafranov (GS) equation to include effects of flow and coupled with the Bernoulli law for density in the direction parallel to the magnetic field. In this presentation, the GGS is compared with the reduced MHD equilibrium equations in the presence of flow for tokamaks, theoretically and numerically.

The GGS equation coupled with the Bernoulli law is a non-linear second-order partial differential equation (PDE) with five free functions for the magnetic flux function. It can be elliptic, hyperbolic or singular depending on the poloidal flow velocity in toroidal geometry. The pressure is not constant on magnetic flux surfaces. Compared with the GS equation for static equilibrium that is an elliptic PDE with two free functions of the magnetic flux function including pressure, the GGS equations is complicated and difficult to solve. To solve numerically, an iteration method with an initial guess of the magnetic flux function is exploited. The Bernoulli law is an algebraic equation for density and solved with a root-finding method with a given magnetic flux function. The GGS with solved with a Poisson solver by substituting the initial guess and the solution of the density. These procedures are iterated until the solution converges. Since the GGS equation is nonlinear, the elliptic PDE for the poloidal flow depends on the solution. To find a solution, the five free functions should be carefully chosen.

By using asymptotic expansions with respect to the inverse aspect ratio of a torus of magnetically confined plasma, reduced equilibrium equations are derived. In Ref. [2], reduced equilibrium equations for high-beta tokamaks are shown. The derivation depends on the order of poloidal flow velocity. Two cases with flow velocity comparable to the poloidal sound velocity and the poloidal Alfven velocity are studied. These are easy to solve compared to the GGS equation and their analytical and numerical solutions are obtained [3-5]. The solutions show qualitative difference in the shift of magnetic flux surfaces and pressure isosurfaces for the cases below and beyond the characteristic velocities.

In this presentation, we will show results of the comparison between the GGS equation and reduced equilibrium equation. To compare the solutions, the equivalent free functions must be used. In GGS equation, free functions are in the form of entropy and enthalpy while, in reduced equations, they are pressure and poloidal Alfven Mach number. The correspondence of these free functions is derived. By using these free functions, benchmark results of numerical calculations will be shown. We have developed a code for the coupled equations of the GGS equation and the Bernoulli law. The GGS equation is solved with the finite element method. The Bernoulli law is solved with the Newton method. Numerical solutions of these equations will be compared with those of the reduced equilibrium equations.

References

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