Linear analysis of magnetorotational instability via reduced four-field model of single helicity and incompressible MHD

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Previous papers [1, 2] studied negative energy modes of magnetorotational instability (MRI) [3, 4]. Non-axisymmetric MRI can have negative energy. The Frieman–Rotenberg equation [5] was used in the previous papers. In this paper, we present a four-field reduced model of single helicity, incompressible magnetohydrodynamics (MHD) that can be used for the MRI analyses.

Let us consider a cylindrical plasma. Physical quantities are assumed to have a single helicity as

$$f(r,\theta,z,t) = \sum_{\ell=-\infty}^{\infty} f_\ell(r,t) \mathrm{e}^{\mathrm{i}\,\ell(M\theta+N\zeta)}, \tag{1}$$

where f is an arbitrary variable, (r,θ,z) are the cylindrical coordinates, $\zeta := z/R_0$ with $2\pi R_0$ being the length of the plasma column, M and N are the principal poloidal and toroidal mode numbers, respectively, and ℓ represents their harmonics. By introducing $\alpha := M\theta/K + z$ with $K := N/R_0$, the phase can be written as $M\theta + N = K\alpha$. Then f depends on r, α and t only.

Let us define an incompressible vector field as

$$\boldsymbol{h} := \frac{1}{K_0^2 r^2}(-Kr\hat{\boldsymbol{\theta}} + M\hat{\boldsymbol{z}}), \tag{2}$$

where $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{z}}$ are unit vectors in the θ and z directions, respectively, and $K_0^2 r^2 := M^2 + K^2 r^2 = 1/|\boldsymbol{h}|^2$. Then $\boldsymbol{h}\cdot\nabla r = \boldsymbol{h}\cdot\nabla\alpha = 0$. By using $\boldsymbol{h}$, an incompressible velocity field $\boldsymbol{u}$ and a magnetic field $\boldsymbol{B}$ are expressed as

$$\boldsymbol{u} = \boldsymbol{h}\times\nabla\varphi(r,\alpha) + u_h(r,\alpha)\boldsymbol{h}, \tag{3}$$
$$\boldsymbol{B} = \nabla\psi(r,\alpha)\times\boldsymbol{h} + B_h(r,\alpha)\boldsymbol{h}. \tag{4}$$

Now, we can derive a four field model by taking the components along $\boldsymbol{h}$ of the vorticity equation, the equation of motion, the Ohm's law, and the induction equation as follows:

$$\frac{\partial U}{\partial t} = -\left[u_h, \frac{1}{K_0^2 r^2} u_h\right] - [\varphi, U] + \left[\varphi, \frac{2MK}{K_0^4 r^4} u_h\right] + \left[B_h, \frac{1}{K_0^2 r^2} B_h\right] + [\psi, J] + \left[\psi, \frac{2MK}{K_0^4 r^4} B_h\right], \tag{5}$$

$$\frac{\partial u_h}{\partial t} = [u_h, \varphi] + [B_h, \psi], \tag{6}$$

$$\frac{\partial \psi}{\partial t} = [\psi, \varphi], \tag{7}$$

$$\frac{\partial B_h}{\partial t} = -K_0^2 r^2 \left[\psi, \frac{1}{K_0^2 r^2} u_h\right] - K_0^2 r^2 \left[\varphi, \frac{1}{K_0^2 r^2} B_h\right] + \frac{2MK}{K_0^2 r^2}[\varphi, \psi], \tag{8}$$

where

$$U := \mathcal{L}\varphi, \tag{9}$$
$$J := \mathcal{L}\psi, \tag{10}$$
$$\mathcal{L} := \frac{1}{Kr}\frac{\partial}{\partial r}\left(\frac{Kr}{K_0^2 r^2}\frac{\partial}{\partial r}\right) + \frac{1}{K^2 r^2}\frac{\partial^2}{\partial \alpha^2}, \tag{11}$$
$$[f,g] := \boldsymbol{h}\cdot\nabla f \times \nabla g. \tag{12}$$

The linear MRI can be studied by linearizing Eqs. (5)–(8). As in the previous studies[1, 2], the plasma is assumed to exist within an annular region between an inner boundary $r = r_1$ and an outer boundary $r = r_2$. The perturbation of φ is assumed to vanish at r_1 and r_2. By assuming the time dependence of perturbed variables as $\mathrm{e}^{-\mathrm{i}\,\omega t}$, we obtain an eigenvalue problem. The equilibrium is given to have a homogeneous ambient magnetic field in the z direction as $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$ with a constant B_0, and a plasma rotation velocity $\boldsymbol{u}_0 = r\Omega(r)\hat{\boldsymbol{\theta}}$ where $\Omega(r) = \Omega_1 r_1^2/r^2$ with a constant Ω_1. Figure 1 shows the real frequency of non-axisymmetric MRI with $M = 1$, $N = 1$ and $\ell = 1$, which successfully reproduced the previous study.

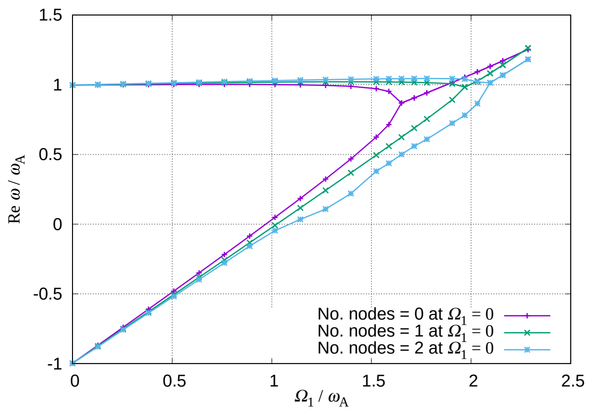

Figure 1: The real frequency of non-axisymmetric MRI with $M = 1$, $N = 1$ and $\ell = 1$.

Acknowledgement

M.F. was supported by JSPS KAKENHI Grand Number JP21K03507 and JP24K06993.

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